

Exam Quantum Physics 2

Date 14 June 2016
Room A. Jacobshal 03
Time 14:00 - 17:00
Lecturer D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are allowed to use the book “Introduction to Quantum Mechanics” by Griffiths
- You are *not* allowed to use print-outs, notes or other books
- The weights of the three exercises are given below
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	8	2a)	6	3a)	10
1b)	8	2b)	8	3b)	10
1c)	8	2c)	10	3c)	6
1d)	8	2d)	8		

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Exercise 1

(a) Consider an electron in the $l = 2$ and $s = 1/2$ state. Use the table below to write down the Clebsch-Gordan decompositions of $|l, s; j, m\rangle = |2, 1/2; 3/2, -3/2\rangle$ and $|l, s; j, m\rangle = |2, 1/2; 3/2, -1/2\rangle$. Explain, without explicit calculations, how one can relate these two decompositions.

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

$1/2 \times 1/2$

1
+1 0
+1/2 +1/2 1 0 0
+1/2 -1/2 1/2 1/2 1
-1/2 +1/2 1/2 -1/2 -1
-1/2 -1/2 1

$1 \times 1/2$

3/2
+3/2 1/2
+1 +1/2 1 +1/2 +1/2
+1 -1/2 1/3 2/3 3/2 1/2
0 +1/2 2/3 -1/3 -1/2 -1/2

2×1

3
+3 2
+2 +1 1 +2 +2
+2 0 1/3 2/3 3 2 1
+1 +1 2/3 -1/3 +1 +1 +1

1×1

2
+2 1
+1 +1 1 +1
+1 0 1/2 1/2 2 1 0
0 +1 1/2 -1/2 0 0 0

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2}\right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$

$2 \times 1/2$

5/2
+5/2 3/2
+2 +1/2 1 +3/2 +3/2
+2 -1/2 1/5 4/5 5/2 3/2
+1 +1/2 4/5 -1/5 +1/2 +1/2

$3/2 \times 1/2$

2
+2 1
+3/2 +1/2 1 +1 +1
+3/2 -1/2 1/4 3/4 2 1
+1/2 +1/2 3/4 -1/4 0 0

$3/2 \times 1$

5/2
+5/2 3/2
+3/2 +1 1 +3/2 +3/2
+3/2 0 2/5 3/5 5/2 3/2 1/2
+1/2 +1 3/5 -2/5 +1/2 +1/2 +1/2

1×1

2
+2 1
+1 +1 1 +1
+1 0 1/2 1/2 2 1 0
0 +1 1/2 -1/2 0 0 0

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle$$

(b) Show explicitly that \vec{J}^2 does not commute with $L_z + 2S_z$ and that \vec{L}^2 does not commute with z .

(c) Consider the effect of a uniform electric field along the \hat{z} direction on the four $n = 2$ levels of hydrogen. Recall that the Stark effect is described by the term $H_S = eEz$ in the Hamiltonian. Consider the matrix elements $\langle n, l', m' | H_S | n, l, m \rangle$ for $n = 2$, which takes the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3eEa & 0 \\ 0 & -3eEa & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

when written on the basis $e_1 = |211\rangle, e_2 = |210\rangle, e_3 = |200\rangle, e_4 = |21-1\rangle$. Provide explanations for the vanishing entries in this matrix.

(d) Explain why hydrogen in the state $(|210\rangle + |200\rangle)/\sqrt{2}$ has a charge distribution with a nonzero electric dipole moment (you may use the result of the previous exercise).

Exercise 2

Consider a two-dimensional square well potential:

$$V(x, y) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \text{ and } 0 \leq y \leq a \\ \infty & \text{elsewhere} \end{cases}$$

(a) The first excited energy level of the system is degenerate. Give this energy and the explicit expressions for the corresponding wave functions.

Next introduce the perturbation:

$$H'(x, y) = \begin{cases} -\alpha\delta(x - y) & \text{for } 0 \leq x \leq a \text{ and } 0 \leq y \leq a \text{ and } \alpha > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b) When using degenerate perturbation theory for the first excited states one generally has to consider off-diagonal matrix elements of H' . Use a symmetry argument to specify a basis on which the specific perturbation H' given above has to be diagonal.

(c) Calculate in first order perturbation theory the correction(s) to the unperturbed energy of the first excited energy level and indicate for which α values the result is expected to be valid. You may make use of the following integral:

$$\int_0^a \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{2\pi x}{a}\right) dx = \frac{a}{4}.$$

(d) Consider the case in which α is time dependent: $\alpha(t) = c\theta(t)$ for some positive constant c . Here $\theta(t)$ is the Heaviside step function (see equation 2.143 of the book). Assume that the system is in the ground state for $t < 0$. Show that in first-order time-dependent perturbation theory the probability to be in the first excited energy level for $t > 0$ vanishes, while for the second excited energy level it is generally nonvanishing (there is no need to calculate that probability explicitly).

Exercise 3

Consider the one-dimensional potential $V(x) = \frac{\hbar^2 \lambda^3}{m} |x|$, with $\lambda > 0$, and consider the following normalized trial wave function for its ground state:

$$\psi_T(x) = \begin{cases} A(b-x) & \text{for } 0 \leq x \leq b \\ A(b+x) & \text{for } -b \leq x \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

with normalization $A = \sqrt{3/2} b^{-3/2}$ and positive parameter b .

(a) Draw the first derivative of the trial wave function and calculate the expectation value of the Hamiltonian for this trial wave function.

(b) Determine the best approximation to the energy of the ground state that one can achieve with the variational method using the given trial wave function. Verify that the answer is indeed larger than the actual ground state energy of $0.81\hbar^2\lambda^2/m$.

(c) Write down a trial wave function that would be sure to give an upper bound on the first excited state.